1　課題と方針

3 次元空間で極座標の変数で表現された関数 $u(r,\theta,\varphi)$ の Laplacian(ラプラシアン):

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

の計算を行いたい. 関数の変数を置き換えて

$$\hat{u}(x,y,z) = u(r(x,y,z), \theta(x,y,z), \varphi(x,y,z))$$

と変換すれば直接計算できるが, ここでは連鎖律を用いて計算を行ない, 最終的に Δu を r,θ,φ と u からなる式として整理する.

2　準備: 極座標の定義他

原点を O に対して任意の点 $P(x,y,z)$ が与えられた時 (必要であれば座標の平行移動をすればよい), OP の距離を $r(r \geq 0)$,z 軸と OP がなす角度を $\theta(0 \leq \theta \leq \pi)$, 点 $P'(x,y,0)$ とした時に x 軸と OP' がなす角度を $\varphi(0 \leq \varphi \leq 2\pi)$ とすると,

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

この定義より以下が導き出される (計算の都合上 $s \equiv \sqrt{x^2+y^2}$):

$$\mathrm{r}^2 = x^2 + y^2 + z^2$$

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{r} = \frac{s}{r}$$

$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z} = \frac{s}{z}$$

$$\tan\varphi = \frac{y}{x}$$

以下に r,θ,φ の 1 階,2 階微分を計算する:

- (準備 1)　$\tan\theta$ の微分:

$$\begin{aligned}\frac{\partial}{\partial\theta}\tan\theta &= \frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{\partial}{\partial\theta}\sin\theta\frac{1}{\cos\theta} + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\cos\theta}\right) \\ &= \frac{\partial}{\partial\theta}\sin\theta\frac{1}{\cos\theta} + \sin\theta\frac{\partial(\cos\theta)}{\partial\theta}\frac{\partial}{\partial(\cos\theta)}\left(\frac{1}{\cos\theta}\right) = \cos\theta\frac{1}{\cos\theta} + \sin\theta(-\sin\theta)\left(-\frac{1}{\cos^2\theta}\right) \\ &= 1 + \tan^2\theta = 1 + \left(\frac{s}{z}\right)^2 = \frac{x^2+y^2+z^2}{z^2} = \frac{r^2}{z^2}\end{aligned}$$

- (準備 2)　$\tan\varphi$ の微分: (前項と同様)

$$\frac{\partial}{\partial\varphi}\tan\varphi = 1 + \tan^2\varphi = 1 + \frac{y^2}{x^2} = \frac{s^2}{x^2}$$

- (準備 3)　s の微分:

$$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{\partial t}{\partial x}\frac{\partial}{\partial t}t^{1/2} = \frac{x}{t^{1/2}} = \frac{x}{s} \qquad ;;\ t \equiv x^2+y^2 \Rightarrow s = \sqrt{t}$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(\frac{1}{s^n}\right) = \frac{\partial s}{\partial x}\frac{\partial}{\partial s}\left(\frac{1}{s^n}\right) = \frac{x}{s}\left(-\frac{n}{s^{n+1}}\right) = -\frac{nx}{s^{n+2}} \qquad ;;\ y \text{ も同様}$$

$$\frac{\partial s}{\partial y} = \frac{y}{s} \qquad ;;\ x \text{ と同様}$$

$$\frac{\partial s}{\partial z} = 0$$

- r の微分:

$$\frac{\partial r}{\partial x}=\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}=\frac{\partial t}{\partial x}\frac{\partial}{\partial t}t^{1/2}=\frac{x}{t^{1/2}}=\frac{x}{r}\qquad ;;\ t\equiv x^2+y^2+z^2\Rightarrow r=\sqrt{t}$$

$$\Rightarrow\quad \frac{\partial}{\partial x}\left(\frac{1}{r^n}\right)=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{r^n}\right)=\frac{x}{r}\left(-\frac{n}{r^{n+1}}\right)=-\frac{nx}{r^{n+2}}\qquad ;;\ y,z\ \text{も同様}$$

$$\Rightarrow\quad \frac{\partial^2 r}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{r}\right)=\frac{1}{r}+x\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\frac{1}{r}-x\frac{x}{r^3}=\frac{r^2-x^2}{r^3}$$

$$\frac{\partial r}{\partial y}=\frac{y}{r}\qquad ;;\ x\ \text{と同様}$$

$$\Rightarrow\quad \frac{\partial^2 r}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{y}{r}\right)=\frac{r^2-y^2}{r^3}$$

$$\frac{\partial r}{\partial z}=\frac{z}{r}\qquad ;;\ x\ \text{と同様}$$

$$\Rightarrow\quad \frac{\partial^2 r}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{z}{r}\right)=\frac{r^2-z^2}{r^3}$$

- θ の微分:

$$\frac{\partial}{\partial x}\tan\theta=\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\tan\theta=\frac{\partial\theta}{\partial x}\frac{r^2}{z^2}$$

$$=\frac{\partial}{\partial x}\left(\frac{s}{z}\right)=\frac{1}{z}\frac{\partial s}{\partial x}=\frac{x}{zs}\quad\Rightarrow\quad \frac{\partial\theta}{\partial x}=\frac{z^2}{r^2}\frac{x}{zs}=\frac{zx}{r^2s}$$

$$\Rightarrow\quad \frac{\partial^2\theta}{\partial x^2}=z\frac{\partial}{\partial x}\left(\frac{x}{r^2s}\right)=z\left\{\frac{1}{r^2s}+\frac{x}{s}\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right)+\frac{x}{r^2}\frac{\partial}{\partial x}\left(\frac{1}{s}\right)\right\}=z\left\{\frac{1}{r^2s}+\frac{x}{s}\left(-\frac{2x}{r^4}\right)+\frac{x}{r^2}\left(-\frac{x}{s^3}\right)\right\}$$

$$=\frac{z(r^2s^2-2x^2s^2-r^2x^2)}{r^4s^3}=\frac{z(r^2y^2-2x^2s^2)}{r^4s^3}$$

$$\frac{\partial}{\partial y}\tan\theta=\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}\tan\theta=\frac{\partial\theta}{\partial y}\frac{r^2}{z^2}$$

$$=\frac{\partial}{\partial y}\left(\frac{s}{z}\right)=\frac{1}{z}\frac{\partial s}{\partial y}=\frac{y}{zs}\quad\Rightarrow\quad \frac{\partial\theta}{\partial y}=\frac{z^2}{r^2}\frac{y}{zs}=\frac{zy}{r^2s}$$

$$\Rightarrow\quad \frac{\partial^2\theta}{\partial y^2}=z\frac{\partial}{\partial y}\left(\frac{y}{r^2s}\right)=z\left\{\frac{1}{r^2s}+\frac{y}{s}\frac{\partial}{\partial y}\left(\frac{1}{r^2}\right)+\frac{y}{r^2}\frac{\partial}{\partial y}\left(\frac{1}{s}\right)\right\}=z\left\{\frac{1}{r^2s}+\frac{y}{s}\left(-\frac{2y}{r^4}\right)+\frac{y}{r^2}\left(-\frac{y}{s^3}\right)\right\}$$

$$=\frac{z(r^2s^2-2y^2s^2-r^2y^2)}{r^4s^3}=\frac{z(r^2x^2-2y^2s^2)}{r^4s^3}$$

$$\frac{\partial}{\partial z}\tan\theta=\frac{\partial\theta}{\partial z}\frac{r^2}{z^2}=\frac{\partial}{\partial z}\left(\frac{s}{z}\right)=-\frac{s}{z^2}\quad\Rightarrow\quad \frac{\partial\theta}{\partial z}=-\frac{z^2}{r^2}\frac{s}{z^2}=-\frac{s}{r^2}$$

$$\Rightarrow\quad \frac{\partial^2\theta}{\partial z^2}=\frac{\partial}{\partial z}\left(-\frac{s}{r^2}\right)=-s\frac{\partial}{\partial z}\left(\frac{1}{r^2}\right)=-s\left(-\frac{2z}{r^4}\right)=\frac{2zs}{r^4}$$

- φ の微分:

$$\frac{\partial}{\partial x}\tan\varphi=\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\tan\varphi=\frac{\partial\varphi}{\partial x}\frac{s^2}{x^2}=\frac{\partial}{\partial x}\left(\frac{y}{x}\right)=-\frac{y}{x^2}\quad\Rightarrow\quad \frac{\partial\varphi}{\partial x}=\frac{x^2}{s^2}\frac{-y}{x^2}=-\frac{y}{s^2}$$

$$\Rightarrow\quad \frac{\partial^2\varphi}{\partial x^2}=\frac{\partial}{\partial x}\left(-\frac{y}{s^2}\right)=-y\frac{\partial}{\partial x}\left(\frac{1}{s^2}\right)=-y\left(-\frac{2x}{s^4}\right)=\frac{2xy}{s^4}$$

$$\frac{\partial}{\partial y}\tan\varphi=\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}\tan\varphi=\frac{\partial\varphi}{\partial y}\frac{s^2}{x^2}=\frac{\partial}{\partial y}\left(\frac{y}{x}\right)=\frac{1}{x}\quad\Rightarrow\quad \frac{\partial\varphi}{\partial y}=\frac{x^2}{s^2}\frac{1}{x}=\frac{x}{s^2}$$

$$\Rightarrow\quad \frac{\partial^2\varphi}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{x}{s^2}\right)=x\frac{\partial}{\partial y}\left(\frac{1}{s^2}\right)=x\left(-\frac{2y}{s^4}\right)=-\frac{2xy}{s^4}$$

$$\frac{\partial}{\partial z}\tan\varphi=\frac{\partial\varphi}{\partial z}\frac{\partial}{\partial\varphi}\tan\varphi=0\quad\Rightarrow\quad \frac{\partial\varphi}{\partial z}=0$$

$$\Rightarrow\quad \frac{\partial^2\varphi}{\partial z^2}=0$$

3 Laplacian の計算

まず x についての 2 階微分を計算すると:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial u}{\partial \varphi}\right) \\
&= \left\{\frac{\partial^2 r}{\partial x^2}\frac{\partial u}{\partial r} + \frac{\partial r}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\right\} + \left\{\frac{\partial^2 \theta}{\partial x^2}\frac{\partial u}{\partial \theta} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\right\} + \left\{\frac{\partial^2 \varphi}{\partial x^2}\frac{\partial u}{\partial \varphi} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \varphi}\right)\right\} \\
&= \frac{\partial^2 r}{\partial x^2}\frac{\partial u}{\partial r} + \frac{\partial^2 \theta}{\partial x^2}\frac{\partial u}{\partial \theta} + \frac{\partial^2 \varphi}{\partial x^2}\frac{\partial u}{\partial \varphi} \\
&+ \frac{\partial r}{\partial x}\left\{\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}\left(\frac{\partial u}{\partial r}\right)\right\} \\
&+ \frac{\partial \theta}{\partial x}\left\{\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right) + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}\left(\frac{\partial u}{\partial \theta}\right)\right\} \\
&+ \frac{\partial \varphi}{\partial x}\left\{\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \varphi}\right) + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \varphi}\right) + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}\left(\frac{\partial u}{\partial \varphi}\right)\right\} \\
&= \frac{\partial^2 r}{\partial x^2}\frac{\partial u}{\partial r} + \frac{\partial^2 \theta}{\partial x^2}\frac{\partial u}{\partial \theta} + \frac{\partial^2 \varphi}{\partial x^2}\frac{\partial u}{\partial \varphi} \\
&+ \left(\frac{\partial r}{\partial x}\frac{\partial r}{\partial x}\right)\frac{\partial^2 u}{\partial r^2} + \left(\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x}\right)\frac{\partial^2 u}{\partial \theta \partial r} + \left(\frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial x}\right)\frac{\partial^2 u}{\partial \varphi \partial r} \\
&+ \left(\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x}\right)\frac{\partial^2 u}{\partial r \partial \theta} + \left(\frac{\partial \theta}{\partial x}\frac{\partial \theta}{\partial x}\right)\frac{\partial^2 u}{\partial \theta^2} + \left(\frac{\partial \theta}{\partial x}\frac{\partial \varphi}{\partial x}\right)\frac{\partial^2 u}{\partial \varphi \partial \theta} \\
&+ \left(\frac{\partial \varphi}{\partial x}\frac{\partial r}{\partial x}\right)\frac{\partial^2 u}{\partial r \partial \varphi} + \left(\frac{\partial \varphi}{\partial x}\frac{\partial \theta}{\partial x}\right)\frac{\partial^2 u}{\partial \theta \partial \varphi} + \left(\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial x}\right)\frac{\partial^2 u}{\partial \varphi^2}
\end{aligned}
$$

y あるいは z についての 2 階微分は上の式で x を (y あるいは z に) 置き換えればよいので

$$
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}
$$

について (r, θ, φ による)u の微分式ごとの係数を r, θ, φ の式として計算する:

- $\dfrac{\partial u}{\partial r}$ の係数:

$$
\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3} + \frac{r^2 - z^2}{r^3} = \frac{3r^2 - r^2}{r^3} = \frac{2}{r}
$$

- $\dfrac{\partial u}{\partial \theta}$ の係数:

$$
\begin{aligned}
\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} &= \frac{z(r^2y^2 - 2x^2s^2)}{r^4s^3} + \frac{z(r^2x^2 - 2y^2s^2)}{r^4s^3} + \frac{2zs}{r^4} = \frac{z(r^2y^2 - 2x^2s^2 + r^2x^2 - 2y^2s^2 + 2s^4)}{r^4s^3} \\
&= \frac{z(r^2s^2 - 2s^4 - 2s^4)}{r^4s^3} = \frac{zr^2s^2}{r^4s^3} = \frac{1}{r^2}\frac{z}{s} = \frac{1}{r^2\tan\theta}
\end{aligned}
$$

- $\dfrac{\partial u}{\partial \varphi}$ の係数:

$$
\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \frac{2xy}{s^4} - \frac{2xy}{s^4} + 0 = 0
$$

- $\dfrac{\partial^2 u}{\partial r^2}$ の係数:

$$
\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2 = \frac{r^2}{r^2} = 1
$$

- $\dfrac{\partial^2 u}{\partial \theta^2}$ の係数:

$$\left(\frac{\partial \theta}{\partial x}\right)^2+\left(\frac{\partial \theta}{\partial y}\right)^2+\left(\frac{\partial \theta}{\partial z}\right)^2=\left(\frac{zx}{r^2 s}\right)^2+\left(\frac{zy}{r^2 s}\right)^2+\left(-\frac{s}{r^2}\right)^2=\frac{z^2x^2+z^2y^2+s^4}{r^4 s^2}=\frac{z^2s^2+s^4}{r^4s^2}=\frac{r^2s^2}{r^4s^2}=\frac{1}{r^2}$$

- $\dfrac{\partial^2 u}{\partial \varphi^2}$ の係数:

$$\left(\frac{\partial \varphi}{\partial x}\right)^2+\left(\frac{\partial \varphi}{\partial y}\right)^2+\left(\frac{\partial \varphi}{\partial z}\right)^2=\left(-\frac{y}{s^2}\right)^2+\left(\frac{x}{s^2}\right)^2+0=\frac{s^2}{s^4}=\frac{1}{s^2}=\frac{1}{r^2}\left(\frac{r}{s}\right)^2=\frac{1}{r^2\sin^2\theta}$$

- $\dfrac{\partial^2 u}{\partial r\partial \theta}, \dfrac{\partial^2 u}{\partial \theta\partial r}$ の係数:

$$\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x}+\frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y}+\frac{\partial r}{\partial z}\frac{\partial \theta}{\partial z}=\frac{x}{r}\frac{zx}{r^2s}+\frac{y}{r}\frac{zy}{r^2s}+\frac{z}{r}\left(-\frac{s}{r^2}\right)=\frac{z(x^2+y^2-s^2)}{r^3s}=0$$

- $\dfrac{\partial^2 u}{\partial \theta\partial \varphi}, \dfrac{\partial^2 u}{\partial \varphi\partial \theta}$ の係数:

$$\frac{\partial \theta}{\partial x}\frac{\partial \varphi}{\partial x}+\frac{\partial \theta}{\partial y}\frac{\partial \varphi}{\partial y}+\frac{\partial \theta}{\partial z}\frac{\partial \varphi}{\partial z}=\frac{zx}{r^2s}\left(-\frac{y}{s^2}\right)+\frac{zy}{r^2s}\frac{x}{s^2}+\left(-\frac{s}{r^2}\right)0=\frac{zxy-zyx}{r^2s^3}=0$$

- $\dfrac{\partial^2 u}{\partial \varphi\partial r}, \dfrac{\partial^2 u}{\partial r\partial \varphi}$ の係数:

$$\frac{\partial \varphi}{\partial x}\frac{\partial r}{\partial x}+\frac{\partial \varphi}{\partial y}\frac{\partial r}{\partial y}+\frac{\partial \varphi}{\partial z}\frac{\partial r}{\partial z}=\left(-\frac{y}{s^2}\right)\frac{x}{r}+\frac{x}{s^2}\frac{y}{r}+0\frac{z}{r}=\frac{-yx+xy}{rs^2}=0$$

以上より Δu の計算を行う; 積の微分の方法を使うと

$$\frac{2}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}=\frac{1}{r^2}\left(2r\frac{\partial u}{\partial r}+r^2\frac{\partial^2 u}{\partial r^2}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$$

あるいは

$$\frac{2}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}=\frac{1}{r}\left\{\frac{\partial u}{\partial r}+\left(\frac{\partial u}{\partial r}+r\frac{\partial^2 u}{\partial r^2}\right)\right\}=\frac{1}{r}\frac{\partial}{\partial r}\left(u+r\frac{\partial u}{\partial r}\right)=\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$$

また

$$\frac{1}{r^2\tan\theta}\frac{\partial u}{\partial \theta}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=\frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial u}{\partial \theta}+\sin\theta\frac{\partial^2 u}{\partial \theta^2}\right)=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)$$

したがって

$$\begin{aligned}\Delta u&=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\\&=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r^2\tan\theta}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}\\&=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}\\&=\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}\end{aligned}$$

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